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Samuel Lelièvre

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Abstract. Abelian differentials on Riemann surfaces can be seen as translation surfaces, which are flat surfaces with cone-type singularities. Closed geodesics for the associated flat metrics form cylinders, whose number under a given maximal length generically has quadratic asymptotics in this length.

Siegel–Veech constants are coefficients of these quadratic growth rates, and coincide for almost all surfaces in each moduli space of translation surfaces. Square-tiled surfaces are some specific translation surfaces whose Siegel–Veech do not equal the generic ones.

It is an interesting question whether, as n tends to infinity, the Siegel–Veech constants of square-tiled surfaces with n tiles tend to the generic constants of the ambient moduli space. Here we prove that it is the case in the moduli space $\mathcal{H}(2)$ of translation surfaces of genus two with one singularity.

Contents

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1. Introduction

1.1. **Geodesics on the torus.** On the standard torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, the number $N(L)$ of families of simple closed geodesics of length not exceeding L is well-known to grow quadratically in L, with

$$N(L) \sim \frac{1}{2\zeta(2)} \cdot \pi L^2$$

which is one half of the asymptotic for the number of primitive lattice points in a disc of radius L. The factor one half comes from counting unoriented rather than oriented geodesics.

By convention, the corresponding *Siegel–Veech constant* is

$$c = \frac{1}{2\zeta(2)}$$

(note that it is the coefficient of πL^2 and not of L^2).

Marking the origin of the torus (i.e. artificially considering it as a singularity or saddle), the number of geodesic segments joining the saddle to itself, of length at most L, coincides with the number of families of simple closed geodesics.

1.2. **Geodesics on translation surfaces.** It is a standard fact that Abelian differentials on Riemann surfaces can be seen as translation surfaces.

On translation surfaces of genus $\geqslant 2$, countings of closed or singular geodesics, similar to those we just described for the torus, can be made.

There, the countings of saddle connections and of families of simple closed geodesics do not coincide, but their growth rates remain quadratic.

Masur proved [Ma88, Ma90] that for every translation surface, there exist positive constants c and C such that the counting functions of saddle connections and of maximal cylinders of closed geodesics satisfy

$$c \cdot \pi L^2 \leqslant N_{\mathrm{cyl}}(L) \leqslant N_{\mathrm{sc}}(L) \leqslant C \cdot \pi L^2$$

for large enough L.

Veech [Ve] proved that on a square-tiled surface (and on any Veech surface) there are in fact *exact quadratic asymptotics* and Gutkin and Judge [GuJu] gave another proof of that. Another proof for the upper quadratic bounds for $N_{\mathrm{cyl}}(L)$ and $N_{\mathrm{sc}}(L)$ was given by Vorobets [Vo].

Eskin and Masur [EM] gave yet another one, and proved that for each connected component of each stratum of each moduli space of normalised abelian (or quadratic) differentials, there are constants c_{sc} and c_{cyl} such that *almost every surface* in the component has $N_{\mathrm{sc}}(L) \sim c_{\mathrm{sc}}\pi L^2$ and $N_{\mathrm{cyl}}(L) \sim c_{\mathrm{cyl}}\pi L^2$.

It is an interesting open problem whether *all* translation surfaces have quadratic growth rates for cylinders of closed geodesics.

The particular constants for many Veech surfaces have been computed explicitly by Veech [Ve], Vorobets [Vo], Gutkin–Judge [GuJu], Schmoll [Schmo], Eskin–Masur–Schmoll [EMS]. The generic constants for the connected components of the strata were computed by Eskin, Masur and Zorich in [EMZ] for the case of abelian differentials.

The particular constants for Veech surfaces usually do not coincide with the generic constants of the strata where they live.

There is also another subtle difference between Veech surfaces and generic surfaces. Define cylinders as **regular** if their boundary components both consist of a single saddle connection. In any connected component of stratum in genus $\geqslant 2$, the counting functions of irregular cylinders are generically subquadratic (in fact a generic surface has no irregular cylinders), while on Veech surfaces they have quadratic asymptotics.

What we will prove however is that individual quadratic constants either for *regular* cylinders or for all cylinders on square-tiled surfaces of the stratum $\mathcal{H}(2)$ (translation surfaces of genus 2 with one singularity) converge as the number of squares tends to infinity to the generic constants of $\mathcal{H}(2)$. See Theorem 1 in § 1.4 for a precise statement.

1.3. **Ratner theory for moduli spaces of abelian differentials.** Analogs of Ratner's theorems classifying invariant measures for the action of unipotent one-parameter groups on homogeneous spaces are expected to hold on strata of the moduli spaces of abelian differentials; the results we prove here could be deduced from such theorems; for the time being, they reinforce the expectation that they do hold.

Some Ratner-like theorems for moduli spaces of abelian differentials have recently been obtained, but do not allow to obtain Theorem 1.

The works of Calta [Ca] and McMullen [Mc] provide a classification of invariant measures in $\mathcal{H}(2)$, albeit for the action of the whole $\mathrm{SL}(2,\mathbf{R})$ and not of unipotent one-parameter subgroups of $\mathrm{SL}(2,\mathbf{R})$.

Eskin, Masur and Schmoll [EMS] have results for the action of unipotent groups on subspaces of $\mathcal{H}(1,1)$.

Eskin, Marklof and Morris [EMWM] have results for the action of unipotent groups on certain moduli spaces of abelian differentials in genus larger than 2.

1.4. **In the stratum $\mathcal{H}(2)$.** In this paper, we are concerned with the stratum $\mathcal{H}(2)$ consisting of abelian differentials in genus 2 with a double zero, or translation surfaces of genus 2 with one singularity (of angle 6π). We prove:

Theorem 1. *Consider a sequence S_n of area* 1 *surfaces in $\mathcal{H}(2)$ such that each surface S_n is tiled by some prime number p_n of square tiles, with $p_n \to \infty$. Then the Siegel–Veech constants for cylinders of closed geodesics on the surfaces S_n tend to $\frac{10}{3} \cdot \frac{1}{2\zeta(2)}$, the generic Siegel–Veech constant of $\mathcal{H}(2)$ for cylinders of closed geodesics. Moreover, the Siegel–Veech constants for regular cylinders also tend to the generic constant, while the Siegel–Veech constants for irregular cylinders tend to* 0.

Remark. We believe that the assumption that the number of squares tiling the surfaces is prime is unnecessary, but we have not yet been able to adapt the calculations to show the convergence of Siegel–Veech constants in the case of nonprime numbers of tiles.

The proof of the theorem relies on fine estimates presented in § 3.1.

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2. Preliminaries

2.1. The stratum $\mathcal{H}(2)$.

2.1.1. *Orbits of square-tiled surfaces.* By a theorem of McMullen [Mc2], in $\mathcal{H}(2)$, for $n > 3$, primitive n-square-tiled surfaces form one orbit E_n if n is even, and two orbits A_n and B_n if n is odd (see [HL1] for the prime n case). Slightly abusing notation, we use the same notation A_n, B_n, E_n for the discrete orbits and for the Teichmüller discs. A formula for the cardinality of E_n (even n) and for the sum of the cardinalities of A_n and B_n is given in [EMS], which in particular results in the asymptotic

$$\frac{3}{8} n^3 \prod_{p|n} (1 - \frac{1}{p^2}).$$

Formulas for the separate countings of A_n and B_n are conjectured in [HL1], which would yield the asymptotics (proved there for prime n):

$$\frac{3}{16} n^3 \prod_{p|n} (1 - \frac{1}{p^2}).$$

Some algebraic properties of the Veech groups are discussed in [HL2].

2.1.2. *Cusps.* Square-tiled surfaces in the stratum $\mathcal{H}(2)$ decompose into either one or two horizontal cylinders, and can be given as coordinates the heights, widths and twist parameters of these cylinders, see [HL1]. Here we are interested in *regular* cylinders of closed geodesics, which exist only in *two-cylinder* decompositions (in one-cylinder decompositions, the unique cylinder has three saddle connections on each boundary component).

The decompositions into cylinders provide a way to parametrise square-tiled surfaces (by the heights, widths and twist parameters of their cylinders). These parameters are very convenient to describe the action of $\mathcal{U} = \{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbf{Z}\}$; it only changes the twist parameters.

The cusps of an $\mathrm{SL}(2,\mathbf{R})$-orbit of square-tiled surfaces, can be identified with the $\mathcal{U}$-orbits of square-tiled surfaces in it, and each cusp has a standard representative (see [HL1, Lemma 3.1]).

In particular, two-cylinder cusps are parametrised by the heights h_i, the widths w_i, and twists parameters t_i of their cylinders ($i \in \{1, 2\}$). A two-cylinder cusp has cusp width $\mathrm{cw}(\mathcal{C}) = \frac{w_1}{h_1 \wedge w_1} \vee \frac{w_2}{h_2 \wedge w_2}$, where $h \wedge w$ denotes the greatest common divisor of h and w, and $a \vee b$ denotes the least common multiple of a and b.

Remark. When the number of tiles is prime, this simplifies to $\mathrm{cw}(\mathcal{C}) = w_1 w_2$.

2.2. **Siegel–Veech constants of cusps.** In the case of the torus, counting families of simple closed geodesics amounts to counting primitive points of $\mathbf{Z}^2$. In this sense, when counting simple closed geodesics of a square-tiled surface of higher genus, we are counting certain multiples of those of the torus.

On a square-tiled surface, as on the torus, the directions which define a decomposition in cylinders of closed geodesics correspond to primitive integer vectors. Better than that, given a primitive square-tiled surface S, each primitive integer vector $(a, b) \in \mathbf{Z}^2$ corresponds to a cusp of the $\mathrm{SL}(2,\mathbf{R})$-orbit of S. Recall that these cusps correspond to $\mathcal{U}$-orbits of square-tiled surfaces in the $\mathrm{SL}(2,\mathbf{R})$-orbit of S.

Here is how to recover the cusp from the primitive integer vector. Since a and b are coprime, by Bezout's theorem, there exist integers c and d such that $ad - bc = 1$. Geometrically, this means (a, b) and (c, d) form an oriented basis of the lattice $\mathbf{Z}^2$. The surface S is tiled by the unit area parallelograms defined by (a, b) and (c, d). Transforming these parallelograms into squares gives a new square-tiled surface. This is done by a linear transformation sending (a, b) to $(1, 0)$ and (c, d) to $(0, 1)$, in other words by applying the matrix $\left(\begin{smallmatrix}a & c\\ b & d\end{smallmatrix}\right)^{-1} \in \mathrm{SL}(2,\mathbf{Z})$.

Of course c and d are not unique, but the various choices of (c,d) give square-tiled surfaces $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)^{-1} \cdot S$ which belong to the same cusp. Quick check: different choices of (c,d) differ by integer multiples of (a,b); accordingly $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} d-kb & -c+ka \\ -b & a \end{smallmatrix}\right)$.

If we count the primitive integer vectors in a ball of radius L which correspond to directions in which S decomposes in cylinders of closed geodesics, we get the same counting function as for the torus.

One thing we could do is to count the primitive integer vectors in a ball of radius L corresponding to a given cusp. The proportion of directions going to different cusp is proportional to their width; see [EMZ, §§ 3.3–3.4 and § 7]. Thus, the asymptotics for each cusp is given by:

$$\frac{\text{width of the cusp}}{\text{sum of the cusp widths of the orbit}} \times \frac{1}{\zeta(2)} \cdot \pi L^2.$$

This is not exactly what we want to count, since we do not want to count the primitive vectors a multiple of which is the holonomy of a cylinder of closed geodesics, but the multiples themselves.

If the corresponding cusp is two-cylinder, with widths w_1, w_2, we want to count the direction not when $\|(a,b)\| < L$ but when $w_1 \cdot \|(a,b)\| < L$.

So the counting for this cusp will have asymptotics

$$\frac{\text{width of the cusp}}{\text{sum of the cusp widths of the orbit}} \cdot \frac{1}{w_1^2} \times \frac{1}{\zeta(2)} \cdot \pi L^2.$$

As a consequence, denoting by D the $\mathrm{SL}(2,\mathbf{Z})$-orbit of S, the counting function for regular cylinders of simple closed geodesics on S has the same asymptotics as

$$\sum_{\text{2-cyl cusps } \mathcal{C}} \frac{\mathrm{cw}(\mathcal{C})}{\#D} \frac{1}{w_1^2} \frac{1}{2\zeta(2)} \pi L^2.$$

If S is a primitive n-square-tiled surface, when we normalise S to area 1, we introduce a factor n in the above asymptotics.

So the asymptotics for the counting function of regular cylinders of simple closed geodesics on a unit area primitive square-tiled surface in an $\mathrm{SL}(2,\mathbf{Z})$-orbit D is given by

$$c(D)\pi L^2$$

and we can write $c(D) = \widetilde{c}(D) \cdot \frac{1}{2\zeta(2)}$ with

$$\widetilde{c}(D) = \frac{n}{\#D} \sum_{\text{2-cyl cusps } \mathcal{C} \text{ of } D} \frac{1}{w_1^2} \mathrm{cw}(\mathcal{C}).$$

3. Asymptotics for a large prime number of squares

Consider some prime n, and an orbit $D_n = A_n$ or B_n. Each cusp is parametrised by some parameters w_1, w_2, h_1, h_2, and twist parameters. By the remark at the end of § 2.1.2, the cusp width is just $w_1 w_2$.

Renaming w_1, w_2, h_1, h_2 as a, b, h, y respectively, the sum over the cusps becomes:

$$\widetilde{c}(D_n) = \frac{n}{\#D_n} \sum_{a,b,h,y} \frac{ab}{a^2}$$

where the sum is over positive integers a, b, h, y satisfying: $a < b$, $ah + by = n$, parity conditions for D_n.

3.1. A simpler sum.

Since $\#D_n$ is, for prime n, asymptotically $\frac{3}{16}n^3$, we first replace $\frac{n}{\#D_n}$ by $\frac{1}{n^2}$.

Second, we momentarily drop the parity conditions; we will reintroduce them in the following subsections.

Last, we drop the condition $a < b$; we will explain later why this does not change the asymptotic.

So we first consider the following simplified sum:

$$S(n) = \sum_{a \geqslant 1} \frac{1}{a^2} \sum_{b \geqslant 1} \sum_{\substack{h \geqslant 1,\, y \geqslant 1 \\ ah+by=n}} \frac{ab}{n^2}.$$

Denote the sum over b by $S(n, a)$. Introducing the variable $m = by$,

$$S(n, a) = \sum_{\substack{1 \leqslant m \leqslant n-a \\ m \equiv n\ [a]}} \sum_{b \mid m} \frac{ab}{n^2} = \frac{a}{n^2} \cdot F(n - a, n, a)$$

where

$$F(x, k, q) = \sum_{\substack{1 \leqslant m \leqslant x \\ m \equiv k\ [q]}} \sum_{b \mid m} b.$$

The following asymptotics hold for $F(x, k, q)$, $S(n, a)$ and $S(n)$.

Lemma 1. *For $k \wedge q = 1$, and $x \to \infty$,*

$$F(x, k, q) = \frac{x^2}{q} \cdot \frac{\pi^2}{12} \prod_{p \mid q} (1 - \frac{1}{p^2}) + O_q(x \log x).$$

Lemma 2.

$$S(n, a) \xrightarrow[n \text{ prime}]{n \to \infty} \frac{\pi^2}{12} \prod_{p \mid a} (1 - \frac{1}{p^2}).$$

Lemma 3. $S(n) \xrightarrow[n \text{ prime}]{n\to\infty} \dfrac{5}{4}$.

Proof of Lemma 1. If m is prime to k, denote by $\overline{m}$ the integer in $\{0,\ldots,q-1\}$ such that $\overline{m}m \equiv 1$ $[q]$, and by $u = u(m,k,q)$ the integer in $\{0,\ldots,q-1\}$ such that $u \equiv \overline{m}k$ $[q]$; error terms depend on q.

$$\begin{aligned}
F(x,k,q) &= \sum_{\substack{1\leqslant md\leqslant x\\ md\equiv k\ [q]}} d \\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \sum_{\substack{1\leqslant d\leqslant x/m\\ d\equiv \overline{m}k\ [q]}} d \\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \sum_{\substack{1\leqslant d\leqslant x/m\\ d\equiv u\ [q]}} d \\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \sum_{1\leqslant u+\lambda q\leqslant x/m} (u+\lambda q) \\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \Big(\Big(\sum_{1\leqslant\lambda\leqslant\frac{1}{q}(\frac{x}{m}-u)} \lambda q\Big) + O(\frac{x}{m})\Big) \\
&= \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \big(\frac{1}{2}q(\frac{x}{qm})^2 + O(\frac{x}{m}) + O(1)\big) \\
&= \frac{x^2}{2q} \sum_{\substack{1\leqslant m\leqslant x\\ m\wedge q=1}} \frac{1}{m^2} + O(x\log x)
\end{aligned}$$

To sum only over the integers m with $m\wedge q = 1$, we can sum over all m with a factor $\mu(m\wedge q)$, so that all terms cancel out except the ones we want.

$$\begin{aligned}
F(x,k,q) &= \frac{x^2}{2q}\sum_{d|q}(\frac{\mu(d)}{d^2}\sum_{m\leqslant x/d}\frac{1}{m^2}) + O(x\log x) \\
&= \frac{x^2}{2q}\sum_{d|q}\frac{\mu(d)}{d^2}(\frac{\pi^2}{6} + O(1/x)) + O(x\log x) \\
&= \frac{x^2}{q}\cdot\frac{\pi^2}{12}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).
\end{aligned}$$

□

Proof of Lemma 2. Lemma 2 follows immediately from Lemma 1 by a dominated convergence argument (similar arguments were used in [HL1, § 7]. □

Proof of Lemma 3. Lemma 3 is a consequence of Lemma 2 by the following observation.

$$\begin{aligned}\sum_{a\geqslant 1}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) &= \prod_{p}(1+\sum_{\nu\geqslant 1}p^{-2\nu}(1-p^{-2\nu})) = \prod_{p}(1+p^2)\\ &= \prod_{p}\frac{1-p^{-4}}{1-p^{-2}} = \frac{\zeta(2)}{\zeta(4)} = \frac{\pi^2/6}{\pi^4/90} = \frac{15}{\pi^2}\end{aligned}$$

□

3.2. **Sums with specified parities.** We introduce sub-sums of $S(n)$ for specified parities of the parameters.

The observation we just made will need to be completed by the following one.

$$\sum_{\substack{a\geqslant 1\\ a\text{ even}}}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \sum_{\substack{a\geqslant 1\\ a\text{ odd}}}\frac{1}{4a^2}\frac{3}{4}\prod_{p|a}(1-\frac{1}{p^2}) + \sum_{\substack{a\geqslant 1\\ a\text{ even}}}\frac{1}{4a^2}\prod_{p|a}(1-\frac{1}{p^2})$$

so that $\displaystyle\sum_{\substack{a\geqslant 1\\ a\text{ odd}}}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \frac{12}{\pi^2}$ and $\displaystyle\sum_{\substack{a\geqslant 1\\ a\text{ even}}}\frac{1}{a^2}\prod_{p|a}(1-\frac{1}{p^2}) = \frac{3}{\pi^2}$.

3.2.1. *Odd widths.* We now consider the sum over odd a and b:

$$S^{\mathrm{ow}}(n) = \sum_{\substack{a\geqslant 1\\ a\text{ odd}}}\frac{1}{a^2}\sum_{\substack{b\geqslant 1\\ b\text{ odd}}}\sum_{\substack{h\geqslant 1, y\geqslant 1\\ ah+by=n}}\frac{ab}{n^2}.$$

We proceed as for the sum $S(n)$: putting

$$F^{\mathrm{ow}}(x,k,q) = \sum_{\substack{1\leqslant m\leqslant x\\ m\equiv k\ [q]}}\sum_{\substack{b|m\\ b\text{ odd}}} b \text{ and } S^{\mathrm{ow}}(n,a) = \frac{a}{n^2}\cdot F^{\mathrm{ow}}(n-a,n,a),$$

$$S^{\mathrm{ow}}(n) = \sum_{\substack{a\geqslant 1\\ a\text{ odd}}}\frac{1}{a^2}S^{\mathrm{ow}}(n,a).$$

The following asymptotics hold for $F^{\mathrm{ow}}(x,k,q)$, $S^{\mathrm{ow}}(n,a)$ and $S^{\mathrm{ow}}(n)$.

Lemma 4. *For odd q, odd k, and $x\to\infty$,*

$$F^{ow}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).$$

For odd a,

$$S^{ow}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{24}\prod_{p|a}(1-\frac{1}{p^2}).$$

Finally,

$$S^{ow}(n) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{1}{2}.$$

Proof.

$$\begin{aligned}
F^{\mathrm{ow}}(x,k,q) &= \sum_{t\geqslant 0}\sum_{\substack{1\leqslant m\leqslant x/2^t\\ 2^t m\equiv k\ [q]\\ m\equiv 1\ [2]}}\sum_{b|m} b\\
&= \sum_{t\geqslant 0}\Bigg(\frac{(x/2^t)^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}(1-\frac{1}{p^2}) + O((x/2^t)\log(x/2^t))\Bigg)\\
&= \frac{x^2}{q}\frac{1}{1-\frac{1}{4}}\frac{\pi^2}{24}(1-\frac{1}{2^2})\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x)\\
&= \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x)
\end{aligned}$$

□

3.2.2. *Odd heights.* We now consider the sum over odd h and y:

$$S^{\mathrm{oh}}(n) = \sum_{a\geqslant 1}\frac{1}{a^2}\sum_{b\geqslant 1}\sum_{\substack{h\geqslant 1, y\geqslant 1\\ h,\ y \text{ odd}\\ ah+by=n}}\frac{ab}{n^2}.$$

Proceeding as previously, we are led to introduce

$$F^{\mathrm{oh}}(x,k,q) = \sum_{\substack{1\leqslant m\leqslant x\\ m\equiv k+q\ [2q]}}\sum_{\substack{b|m\\ m/b \text{ odd}}} b \text{ and } S^{\mathrm{oh}}(n,a) = \frac{a}{n^2}\cdot F^{\mathrm{oh}}(n-a,n,a),$$

$$\text{and to write } S^{\mathrm{oh}}(n) = \sum_{a\geqslant 1}\frac{1}{a^2}S^{\mathrm{oh}}(n,a).$$

The following asymptotics hold for $F^{\mathrm{oh}}(x,k,q)$, $S^{\mathrm{oh}}(n,a)$ and $S^{\mathrm{oh}}(n)$.

Lemma 5. *For even q, odd k, and $x\to\infty$,*

$$F^{oh}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).$$

For odd q, odd k, and $x \to \infty$,

$$F^{oh}(x,k,q) = \frac{x^2}{q}\frac{\pi^2}{32}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).$$

For even a,

$$S^{oh}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{24}\prod_{p|a}(1-\frac{1}{p^2}).$$

For odd a,

$$S^{oh}(n,a) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{\pi^2}{32}\prod_{p|a}(1-\frac{1}{p^2}).$$

Finally,

$$S^{oh}(n) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{1}{2}.$$

Proof. For even q and odd k:

$$\begin{aligned}
F^{\mathrm{oh}}(x,k,q) &= \sum_{\substack{1\leqslant m\leqslant x\\ m\equiv k+q\ [2q]}} \sum_{b|m} b \\
&= \frac{x^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}(1-\frac{1}{p^2}) + O(x\log x) \\
&= \frac{x^2}{q}\frac{\pi^2}{24}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).
\end{aligned}$$

For odd q and odd k:

$$\begin{aligned}
F^{\mathrm{oh}}(x,k,q) &= \sum_{t\geqslant 1} \sum_{\substack{1\leqslant m\leqslant x/2^t\\ 2^t m\equiv k+q\ [2q]\\ m \text{ odd}}} \sum_{b|m} 2^t b \\
&= \sum_{t\geqslant 1} 2^t \sum_{\substack{1\leqslant m\leqslant x/2^t\\ 2^{t-1} m\equiv \frac{k+q}{2}\ [q]\\ m \text{ odd}}} \sum_{b|m} b \\
&= \sum_{t\geqslant 1} 2^t \frac{(x/2^t)^2}{2q}\frac{\pi^2}{12}\prod_{p|2q}(1-\frac{1}{p^2}) + O(x\log x) \\
&= \sum_{t\geqslant 1} \frac{1}{2^t}\frac{x^2}{q}\frac{\pi^2}{24}(1-\frac{1}{2^2})\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x) \\
&= \frac{x^2}{q}\frac{\pi^2}{32}\prod_{p|q}(1-\frac{1}{p^2}) + O(x\log x).
\end{aligned}$$

$\square$

3.2.3. *Mixed parities.* Dealing with the even-odd sums as above would be most cumbersome; this is fortunately not necessary. Indeed, since $S(n) = S^{\mathrm{ow}}(n) + S^{\mathrm{oh}}(n) + S^{\mathrm{eo}}(n)$, and we know the limits of $S(n)$, $S^{\mathrm{ow}}(n)$ and $S^{\mathrm{oh}}(n)$ when n tends to infinity staying prime, we have:

$$S^{\mathrm{eo}}(n) \xrightarrow[n \text{ prime}]{n\to\infty} \frac{1}{4}.$$

3.3. **Asymptotics for orbits A and B.** We end by showing that the limit we obtained is unchanged by adding a condition $a < b$.

Indeed, since $\#\{(h,y)\colon h \geqslant 1,\ y \geqslant 1,\ ah + by = n\} \leqslant n$, the sum $\displaystyle\sum_{b=1}^{a} \sum_{\substack{h\geqslant 1,\, y\geqslant 1\\ ah+by=n}} \frac{ab}{n^2}$ is $O(1/n)$, where the constant of the O depends on a.

This also shows that the constants for irregular cylinders tend to 0.

Putting things together, $\widetilde{c}(A_n)$ and $\widetilde{c}(B_n)$ have the same asymptotics as $S^{\mathrm{A}}(n) = \frac{16}{3}(S^{\mathrm{oh}}(n) + \frac{1}{2}S^{\mathrm{eo}}(n))$ and $S^{\mathrm{B}}(n) = \frac{16}{3}(S^{\mathrm{ow}}(n) + \frac{1}{2}S^{\mathrm{eo}}(n))$, so they both tend to $\frac{10}{3}$.

4. Concluding remarks

Numerical evidence suggests that the convergence to the generic constants of the stratum occurs not only for prime n but for general n; however a proof, relying on the validity of the conjecture on separate countings by orbits in the case of odd numbers of squares, would involve some complications in the calculations which would make the exposition tedious.

A similar study for the quadratic constants that appear in the counting of saddle connections could also be made. There one has to take into consideration both one-cylinder and two-cylinder cusps, and some interesting phenomena can be observed: numerical calculations suggest that the sum of the contributions of one-cylinder and two-cylinder cusps has a limit, but separate countings for one-cylinder cusps do not have a limit for general n; their asymptotics have fluctuations involving the prime factors of n.

References

[Ca] K. Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.* **17** (2004), 871-908.

[EMWM] A. Eskin, J. Marlkof, D. Witte Morris. Unipotent flows on the space of branched covers of Veech surfaces. Preprint (2004).

[EM] A. Eskin, H. Masur. Asymptotic formulas on flat surfaces. *Ergodic Theory Dynam. Systems* **21**:2 (2001), 443–478.

[EMS] A. Eskin, H. Masur, M. Schmoll. Billiards in rectangles with barriers. *Duke Math. J.* **118**:3 (2003), 427–463.

[EMZ] A. Eskin, H. Masur, A. Zorich. Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel–Veech constants. *Publ. IHES* **97**:1 (2003), 61–179.

[GuJu] E. Gutkin, C. Judge. Affine mappings of translation surfaces: geometry and arithmetic. *Duke Math. J.* **103**:2 (2000), 191–213.

[HL1] P. Hubert, S. Lelièvre. Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$. To appear in *Isr. J. Math.*

[HL2] P. Hubert, S. Lelièvre. Noncongruence subgroups in $\mathcal{H}(2)$. *Internat. Math. Res. Notices* **2005**:1 (2005), 47–64.

[Ma88] H. Masur. Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential. *Holomorphic functions and moduli*, vol. 1, D. Drasin, ed., Springer-Verlag, Berlin (1988), 215–228.

[Ma90] H. Masur. The growth rate of trajectories of a quadratic differential. *Ergodic Theory Dynam. Systems* **10** (1990), 151–176.

[Mc] C. T. McMullen. Dynamics of $\mathrm{SL}_2\,\mathbf{R}$ over moduli space in genus two. Preprint (2003).

[Mc2] C. T. McMullen. Teichmüller curves in genus two: discriminant and spin. Preprint (2004).

[Schmo] M. Schmoll. On the asymptotic quadratic growth rate of saddle connections and periodic orbits on marked flat tori. *Geom. Funct. Anal.* **12**:3 (2002), 622–649.

[Ve] W. A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.* **97**:3 (1989), 553-583.

[Vo] Ya. B. Vorobets. Ergodicity of billiards in polygons. *Mat. Sb.* **188**:3 (1997), 65–112.